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Isomorphism problem of endomorphisms

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§1. *Introduction.* D. S. Ornstein and others got remarkable results in the isomorphism problem of automorphisms. They proved that two Bernoulli shifts with the same entropy are isomorphic and obtained the conditions for automorphisms to be Bernoulli. But in the case of endomorphisms the circumstances are different, for example, two Bernoulli endomorphisms (=one-sided Bernoulli shifts) are not always isomorphic (except trivial cases) even if their entropies are the same. We obtained some results in the isomorphism problem of endomorphisms [1, 2, 3] motived by the innovation problem in the theory of stationary processes [4] and the existence problem of Bernoulli generators for number-theoretic transformations.

In this talk we summarize the results in [1, 2, 3] without proofs.

§2. *Definitions.* Let $(X, \mathfrak{F}, \mu)$ denote a non-atomic Lebesgue probability space, and T be an endomorphism of X (i.e. $T^{-1}\mathfrak{F} \subset \mathfrak{F}$, and $\mu(T^{-1}A)=\mu(A)$ for all $A \in \mathfrak{F}$).

An endomorphism T of $(X, \mathcal{F}, \mu)$ is called *Bernoulli* if there exists a measurable partition P of X such that

$$\text{(i)} \quad \bigvee_{0}^{\infty} T^{-n}P = \varepsilon \text{ , the partition into the individual points,}$$

(P is a *generator* of T)

and (ii) $\mu(A | \bigvee_{1}^{\infty} T^{-n}P) = \mu(A)$ a.e. for all $A \in \mathcal{F}(P)$

(P is a *B-partition* of T).

If there exists a measurable partition P (with n-~~atoms~~ cells, $2 \leq n \leq \infty$) of X satisfying (i) and

$$\text{(iii)} \quad \mu(A | \bigvee_{1}^{\infty} T^{-n}P) = \mu(A | T^{-1}P) \quad \text{a.e. for all } A \in \mathcal{F}(P),$$

(P is a *M-partition* of T)

then T is called an $(n\times n)$ *Markov endomorphism*.

In these cases P is called a *B-generator* (*M-generator*, respectively) of T.

Two endomorphisms T_i of $(X_i, \mathcal{F}_i, \mu_i)$, $i=1,2$, respectively are called *isomorphic* if there exists an isomorphism (mod. 0) ϕ from X_1 onto X_2 such that $\phi \circ T_1 = T_2 \circ \phi$.

§3. *Isomorphism theorem for Bernoulli endomorphisms.* [1]

We say that two probability distributions are of the *same type* if there exists a 1-1 correspondence between their atoms (point masses) such that the corresponding atoms have the same measure.

Theorem 1. Let T_i be Bernoulli endomorphisms of $(X_i, \mathcal{F}_i, \mu_i)$ with B-generators P_i, i=1,2, respectively. Then they are isomorphic if and only if the distributions of P_1 and P_2 are of the same type.

This theorem is essentially reduced to the following

Lemma 1. Let T be an endomorphism of $(X, \mathcal{F}, \mu)$ with B-generator Q and a generator P. Then the distribution of P with respect to the conditional measure $\mu(\cdot|r)$ has the same type as that of Q for a.e. $r \in \bigvee_1^\infty T^{-n}P$.

We will now discuss the uniqueness of B-generators of T. Let T be an endomorphism of $(X, \mathcal{F}, \mu)$ and put

$$\mu_T(x) = \mu(\{x\}|\ C_{T^{-1}\varepsilon}(x))$$

where $C_{T^{-1}\varepsilon}(x)$ denotes the cell of the partition $T^{-1}\varepsilon$ containing x. Note that if T has a B-generator $P=\{p_i\}$, then $\mu_T(x)=\mu(p_i|C_{T^{-1}\varepsilon}(x))=\mu(p_i)$ a.e. $x \in p_i$. Since the function μ_T is measurable, we can define a measurable partition R_T of X generated by μ_T and call it the *proper partition* of T.

Theorem 2. If T has a countable B-generator with distinct probabilities, then R_T is the unique B-generator of T.

§4. *A necessary condition for an endomorphism to be Bernoulli.*

By Lemma 1 the following condition (U) is necessary for an endomorphism T on $(X, \mathcal{F}, \mu)$ with a generator $P=\{p_i\}$ to be countable Bernoulli :

(U) *There exists a probability vector* $\rho=(\rho_1, \rho_2, \cdots)$

with positive ρ_i*'s such that* $\mu_T(x)=\mu(p_i \mid C_{T^{-1}\varepsilon}(x))=\rho_i$ a.e. $x \in p_i$.

In the case that T is a Markov endomorphism with a M-generator P satisfying (U), (T, P) is called *uniform* (see [4]).

On the other hand we can prove the following

Lemma 2. If an endomorphism T on $(X, \mathcal{F}, \mu)$ with a generator $P=\{p_i\}$ satisfies the condition (U) with distinct ρ_i's, then the proper partition R_T is a B-partition of T.

(This is proved in [1] in the case (T, P) is Markovian)

§5. *Number-theoretic transformations.* [2]

Now we will introduce three well-known endomorphisms in $(0,1]$ or $[0,1)$:

$Tx = \{1/x\}$, $x \in (0,1]$ (*continued-fraction*),

$Tx = \{\beta x\}$, $x \in [0,1)$ (β-*expansion*), $\beta>1$,

$Tx = \{\beta x+\alpha\}$, $x \in [0,1)$ (*linear mod.* 1), $\beta \geqq 2$, $\alpha \in (0,1)$,

where $\{y\}$ denotes the fractional part of y.

Only checking the necessary condition stated in §3 (Lemma 1) for these transformations, we obtain the following

Theorem 3. Continued-fraction trasformation is neither Bernoulli, nor countable Markov. β-expansion transformations and linear mod. 1 transformations are not Bernoulli except the case of integer β, and if β is an integer these are Bernoulli and isomorphic.

§6. *A sufficient condition for R_T to generate* (Markovian case).[1]

It seems difficult to find some conditions for R_T to be a generator of T in a general situation, so we will restrict ourselves to the Markovian case.

Let T be an endomorphism with a M-generator $P=\{p_i\}$ and $\Pi=(\pi_{i,j}\ ;\ i,j=1,2,\cdots)$ denote the trasition matrix defined by

$$\pi_{i,j} = \mu(\, p_j \mid T^{-1}p_i\,) \qquad i,j = 1,\, 2,\, \cdots ,$$

and $\pi^{(i)}=(\pi_{i,1},\, \pi_{i,2},\, \cdots)$ the i-th row vector of Π.

Assume now T and P satisfy the condition (U) in §4 with distinct ρ_i's. Then given k and i we get the unique j such that $\pi_{k,j}=\rho_i$, so we can define

$$\phi(\,k\ ;\ i\,) = j \quad \text{if} \quad \pi_{k,j}=\rho_i ,$$

and $$\phi(k\ ;\ i_n,\cdots,\, i_1,\, i_o)=\phi(\phi(k\ ;\ i_n,\cdots,\, i_1)\ ;\ i_o)$$

inductively. Using this notation we have

Theorem 4. Let T be an endomorphism with M-generator $P=\{p_i\}$. In order that T has a countable B-generator of which ~~atoms~~ cells have all distinct measures, it is necessary and suffucient that P satisfies the condition (U) with distinct ρ_i's and also the condition

(P) *for any* $\delta>0$ *there exist* $j_o,\, m,\, i_m,\cdots,\, i_o$ *such that*

$$\sum_{k:\ \phi(k;i_m,\cdots,\, i_o)=j_o} \mu(p_k) > 1-\delta\ .$$

(point-collapsing condition in the terminology of [4])

Remark. The assumption of distinctness of ρ_i's is removable, so we have Theorem 4 without the underlined parts. For example, the isomorphism between a Markov $\begin{bmatrix} a & b & a \\ b & a & a \\ a & a & b \end{bmatrix}$ and a Bernoulli $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ endomorphisms is just obtained as the isomorphism between $\begin{bmatrix} a & b & c \\ b & c & a \\ a & c & b \end{bmatrix}$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

§7. *Isomorphism theorems between Markov endomorphisms.* [3]

First we will consider general Markov endomorphisms with countable M-generators.

Theorem 5. Let T and S be ergodic Markov endomorphisms with transition matrices $\Pi=(\pi_{i,j})$ and $\Gamma=(\gamma_{i,j})$ respectively. Then T and S are isomorphic if and only if there exists a measurable, integer-valued function $\eta(x)$ such that

(i) $\gamma_{\eta(Tx),\xi(x)} = \pi_{\xi(Tx),\xi(x)}$ a.e.

(ii) $Q=\{\ q_j=\{x:\ \eta(x)=j\}\ :\ j=0,1,2,\cdots\}$ is a M-generator of T,

where $\xi(x)=j$ if $x\in p_j$ and $P=\{p_j\}$ is a M-generator of T with Π.

Now let $\pi^{(i)}$ be the i-th row vector of Π and consider the condition

(D) *the distributions of* $\pi^{(i)}$, $i=0,1,2,\cdots$ *are all different.* Then we have

Theorem 6. Two ergodic Markov endomorphisms with M-generators satisfying the condition (D) are isomorphic, if and only if their transition matrices are the same except the numbering of cells of M-generators.

Next we will study uniform Markov endomorphisms which are not point collapsing. We put the condition

(G) *there exists a finite M-generator of which common distribution* $\rho=(\rho_i\ ;\ 0\leqq i\leqq N-1)$ *consists of distinct* ρ_i*'s, and its transition matrix* $\Pi=(\pi_{i,j}\ ;\ 1\leqq i,j\leqq N-1)$ *satisfies* $\pi_{i,j} = \rho_{i+tj+t_0}$, $0\leqq i,j\leqq N-1$, *for some* $0\leqq t,t_0\leqq N-1$ *such that* $(N,t)=1$, *where the addition is taken to be mod. N.*

Theorem 7. Let T and S be Markov endomorphisms with transition matrices $\Pi=(\pi_{i,j}\ ;\ 0\leqq i,j\leqq N-1)$ and $\Gamma=(\gamma_{i,j}\ ;\ 0\leqq i,j\leqq N-1)$ respectively, which satisfy the condition (G) with the same common distribution $\rho=(\rho_i\ ;\ 0\leqq i\leqq N-1)$ (i.e. $\pi_{i,j}=\rho_{i+tj+t_0}$, $\gamma_{i,j}=\rho_{i+sj+s_0}$ for $0\leqq i,j\leqq N-1$, and $(N,t)=(N,s)=1$). Then T and S are isomorphic if and only if $t=s$ and s_0-t_0 is a multiple of $(t+1, N)$.

Using these theorems (Theorem 5, 6, 7 and Theorem 4) we can classify 3×3 Markov endomorphisms completely, but we have no space to write down it. (See the examples stated in [3])

References

[1] I. Kubo, H. Murata and H. Totoki, On the isomorphism problem for endomorphisms of Lebesgue spaces, I, *to appear in* Publ. R.I.M.S. Kyoto Univ. 9, No.2(1973).

[2] ______________________, II, ibid.

[3] ______________________, III, ibid.

[4] M. Rosenblatt, Stationary processes as shifts of functions of independent random variables, J. Math. Mech. 8(1959), 665-681.